Master's Thesis :

# Stability of Nonlinear Systems of Ordinary Differential Equations Using Lyapunov Functions

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# Stability of Nonlinear Systems of Ordinary Differential Equations Using Lyapunov Functions


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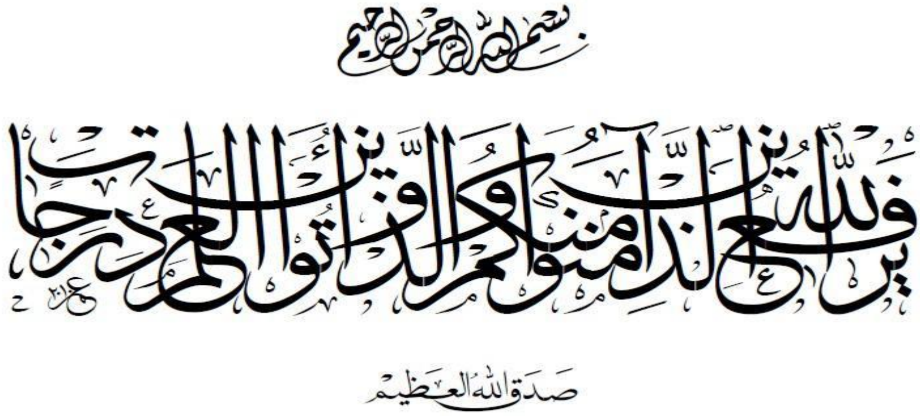


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## Dedication

To the souls of my mother and father, whose blessing, laughter, and prayers still light my path despite their absence. May God have mercy on you and grant you Paradise.

To my husband, my greatest support and strength in every step. Thank you for your constant presence, which gives my heart strength and my days meaning.

To my beloved children, who have borne with me and accompanied my efforts and hardships with patience and love. You are my greatest motivation, and with you, my joy in this achievement is complete.

Sawsan

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Researcher

# Contents

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# Abstract

In this research, we study the stability of nonlinear systems of first order ordinary differential equations with constant coefficients using Lyapunov functions. For that we study the stability of these systems in general, then we give some basic concepts about the equilibrium point and types of stability according to Lyapunov, in addition to some Lyapunov theorems which help us to analysis the stability using Lyapunov Functions. Using this methodology, we can determine system behavior without requiring explicit solutions to the systems. We apply this methodology to analysis the stability of a drone in three-dimensional space (3D-drone) using a nonlinear mathematical model. This model incorporates translational motion along the three axes, as well as the rotational angles around each axis. we give some examples to illustrate our study.

# Introduction

Nonlinear ordinary differential equations ($NLODE_s$) are fundamental to the mathematical modeling of many real-world phenomena. These equations naturally appear in various scientific and engineering fields, such as physics, mechanical, electrical engineering, weather forecasting, population dynamics, economics, and many other natural processes. While traditional analytical methods can handle some types of $NLODE_s$, obtaining explicit solutions for nonlinear systems remains a significant challenge. In many cases, solutions can only be expressed as series or approximations, necessitating alternative approaches to understand the behavior of these systems. The difficulties associated with solving $NLODE_s$ have led to the development of qualitative analytical methods that aim to study the properties of systems without relying on explicit solutions. The true origins of this trend can be traced back to the work of the French mathematician "Henri Poincare", whose contributions laid the theoretical foundation for the science of nonlinear dynamical systems.

Although Linear differential equations ($LODE_s$) developed rapidly during the 18$^{th}$ century, pinpointing the exact period of emergence and study of $NLODE_s$ remains more complex due their gradual development across different eras, their diversity and the difficulty of solving them.

Among the most important aspects related to the study of nonlinear systems is the concept of stability, a pivotal concept in these systems, especially control systems. Stability is based on the system's ability to maintain its behavior near its equilibrium point or return to it in the event of minor disturbances. The earliest concept of stability arose from the study of the equilibrium of dynamic systems. Some of the earliest contributions date bake to 1644 when "Torricelli" studied the equilibrium of rigid bodies under the influence of gravity, followed by fundamental results presented by "Lagrange" in 1788 regarding the stability of conservative dynamic systems.

Among the most influential and effective tools for analyzing the stability of nonlinear and dynamic systems is "Lyapunov's theory", which is considered one of the most prominent tools that revolutionized the analysis of system stability, particularly in nonlinear systems. Lyapunov presented our strong mathematical approach based on the construction of special functions known as "Lyapunov functions", which enable the study of the stability of systems without the need to directly solve differential equations. They are standard functions used to determine the nature of the system's behavior near equilibrium points. This theory has become one of the most important pillars in the analysis of nonlinear systems and the design of modern control systems, and a starting point for many developments in the field of stability and control.

There are many papers studied stability of linear and nonlinear systems of $ODEs$ of first order. In 2012, Thnoun *et al.* [37] studied the stability of a periodic motion for physical application which is leads to differential equations of second order (Double and Spherical Pendulum) respectively using the stability of equilibrium position given by Lyapunov and Ghetagev's method, which depends on principle of energy conservation. Also, he described periodic motion and explain the phase plane and state of the stability for double and spherical pendulum using (Maple). In 2015, Morgan [18] used the linearization techniques and linear differential equation theory to analyze nonlinear $ODEs$, he took a special way to analyze the solutions of the nonlinear systems of $ODEs$. Also, he provide stability analysis, phase portraits, and numerical solutions for these systems. In 2020, Al-Zenati *et al.* [33] studied nine cases of solutions stability of $ODEs$ at the critical point zero according to the types of the roots of the equation. In 2021, Sivaram [24] presented differential methods to find the solutions of linear systems of $ODEs$ with constant coefficients and compared the difficulties of each method. It was observed that the application of matrix methods are very useful in discussing the stability of a dynamical system with constant coefficients. In 2024, Al-Guhus [35] studied the stability analysis of dynamic systems using a set of complex mathematical and numerical ways to investigate the effect of time step size and initial parameters on the accuracy and stability of numerical solutions.

The main aim of this research is to study the stability of nonlinear systems of first order $ODEs$ with constant coefficients using Lyapunov functions, and apply it to the stability analysis of a drone in three-dimensional space. By employing this methodology, we can determine system behavior without requiring explicit solutions to the systems.

This research will be organized as the following:

**Chapter 1: Nonlinear Systems of First Order Ordinary Differential Equations with Constant Coefficients**

In this chapter, we give important basic concepts about the nonlinear systems of first order $ODEs$ and some of methods to solve it.

**Chapter 2: Stability of Nonlinear Systems of First Order Ordinary Differentinial Equations**

In this chapter, we study the stability of nonlinear systems of $ODEs$ of first order with constant coefficients. For that, we give the basic concept of stability, then analysis the stability of these systems.

**Chapter 3: Using Lyapunov Functions for Studing the Stability of Nonlinear Systems of First Order Ordinary Differential Equations**

In this chapter, we study stability of nonlinear systems of $ODEs$ of first order at the equilibrium point using Lyapunov functions. All basic concepts of the stability in this chapter are in the sense of Lyapunov.

**Chapter 4: Application of Stability Analysis of a 3D-Drone**

In this study, we use Lyapunov functions for the stability analysis of a drone in three-dimensional space using a nonlinear mathematical model.

# Chapter 1

# Nonlinear Systems of First Order Ordinary Differential Equations with Constant Coefficients

# Nonlinear Systems of First Order Ordinary Differential Equations with Constant Coefficients

## 1.1 Introduction

The nonlinear systems of differential equations have a fundamental place in the study of applied mathematics, dynamical systems, complex geometric phenomena and natural phenomena. They represent a mathematical model for many application such as physics, engineering, and the life sciences.

The first order ordinary differential equation is an equation that relates between unknown function $x(t)$ of independent variable $t$ and its first derivative $\dot{x}(t)$. It is written in the form:

$$\dot{x}(t) = f(x, t).$$

A system of ordinary differential equations is a system consisting of a set of equations, each equation consisting unknown function and its derivatives in one variable. A nonlinear first order ordinary differential equation is an equation containing a first order derivative of the dependent variable only with respect to the independent variable, it is written on the following form:

$$f(t, x, \dot{x}) = 0, x = x(t),$$

where $t$ is the independent variable, $x(t)$ is the dependent variable, $\dot{x} = \frac{dx}{dt}$ is the first derivative, and $f$ is a nonlinear function. An ordinary differential equation is called nonlinear if it satisfies one of the following conditions:

i. The appearance of $x$ or $\dot{x}$ raised to a power other than integer one. For example: $x^2, \sqrt{x}, x^3$ .

ii. The appearance of $x$ or $\dot{x}$ inside a nonlinear function, such as: $sin(x)$, $cos(\dot{x})$, $In(x), e^x, \dots$ . For example: $\dot{x} + sin(x) = t$.

iii. The existence of a multiplication between $x$ and$\dot{x}$. For example: $\dot{x}x = t$.

In this chapter, we give some definitions and basic concepts about the nonlinear system of ordinary differential equation with constant coefficients. Throughout this chapter, we refer to the ordinary differential equation by the symbol $ODE$, the nonlinear ordinary differential equation by the symbol $NLODE$ and the nonlinear first order ordinary differential equation by $1^{st} - NLODE$.

## 1.2 Basic Definitions

It this section, we give some basic definitions related by the nonlinear system of $ODE_S$ .

### 1.2.1 The Order of the System of $ODEs$ [36]

It is the highest derivative that appears in all equations in the system. For example the following system:

$$\dot{x_1} = 2x_1 + x_1 x,$$

$$\dot{x_2} = 3x_2 + (x_1)^2,$$

is of the first order, but the following system:

$$\dot{x_1} - 2x_2 + sin(x_1) = 0,$$

$$\dddot{x_2} + x_1 + x_1 x_2 = 0,$$

is of the third order.

### 1.2.2 The Degree of the System of $ODEs$ [36]

It is the exponent of the highest derivatives which appears in the equations of that system. For example, the following system

$$\dot{x_1} = 3(x_2)^2 + cos(x_1),$$

$$\dot{x_2} = 2x_2 + (x_1 x_2)^2,$$

is of the first degree. But the following system:

$$(\dot{x_1})^3 = 5x_2 + x_1,$$

$$\dot{x_2} = 2x_1 + x_1x_2,$$

is of the third degree.

### 1.2.3 The Nonlinear System of $ODEs$ [3]

The system of $ODE_s$ is called nonlinear if at least one of its equations is nonlinear. For example, the following system is nonlinear:

$$\dot{x_1} + x_1x_2 - x_1^2 = 0,$$

$$(\ddot{x_2})^2 + sin(x_2) + 2x_1 = 0.$$

The system of $ODE_s$ is called linear if every equation in it is a linear. In other words, if it satisfies the following:

i. The dependent variable and all its derivatives are of the first degree.
ii. The dependent variable and its derivatives are not multiplied by each other.

For example the following system:

$$3\ddot{x_1} + 4x_1 = 0,$$

$$\dot{x_2} + 5x_1 - 7x_2 = 0$$

is a linear system. Note that, the last two systems in examples are with constant coefficients.

### 1.2.4 The Homogeneous and Nonhomogeneous System [20]

Let us consider the following $NLODE$:

$$a_0(t)(y^{(n)})^{m_0} + a_1(t)(y^{(n-1)})^{m_1} + \cdots + a_{n-1}(t)(\dot{y})^{m_{n-1}} + a_n(t)(y)^{m_n} = f(t), \quad (1.2.1)$$

where $a_0(t), a_1(t), \dots, a_n(t)$ are constant coefficients such that $a_0(t) \neq 0$, or functions in a variable $t$, $f(t)$ is known function defined in $t$, $a_0(t) \neq 0$, and $m_0$, $m_1, \dots, m_n$ are integers more than 1. The equation (1.2.1) is called a homogeneous $LODE$ if $f(t) = 0$ and every term in the equation contains the independent variable $t$. If $f(t) \neq 0$, then the equation (1.2.1) is called a nonhomogeneous. Hence the system is homogeneous if every equation in it is homogeneous, otherwise the system is nonhomogeneous. For example, the following system:

$$(\dot{x_1})^3 + 5x_2 + x_1 = 0,$$

$$\dot{x_2} + 2x_1 + x_1x_2 = 0,$$

is homogeneous, but the following system:

$$\dot{x_1} + 2x_1 + sin\,(x_2) = 0,$$

$$\dot{x_2} + x_2^2 = e^{x_1},$$

is nonhomogeneous.

### 1.2.5 The Initial Conditions and Boundary Conditions [14]

**First: Initial Conditions**

By $ODEs$, we can describe many natural and geometric phenomena. Suppose $f(t, x, \dot{x}, \dots x^{(n)}) = 0$ is an $n^{th} - ODE$. This equation usually has general solution contains $n$-arbitrary constants on an open interval $I$ containing $x = x_0$. to determine the particular solution from the general solution, there must give $n$ initial conditions at the initial moment $x = x_0$ : $x(t_0), \dot{x}(t_0), \ddot{x}(t_0), \dots, x^{(n-1)}(t_0)$.

If we have system of $n$ of $1^{th} - NLODEs$ and dependent variables: $x_1(t), x_2(t) \dots x_n(t)$, then we need one initial condition for each dependent variable: $x_1(0), x_2(0) \dots x_n(0)$. So each equation in the system has an associated initial value; this is called the unique solution. Note that, the system of $n$ of $1^{th} - NLODEs$ with initial conditions is called initial value problem (IVP).

**Second: Boundary Conditions [19]**

In many practical applications, solving $ODEs$ does not specify the solution requirements at singular point in time as an initial condition. Thus, instead of determining the value of function $x(t)$ and its derivatives at single value of $t$ such as $t_0$ , conditions are imposed at different values of $t$, such as $t = a$ and $t = b$.

### 1.2.6 The General Solution and Particular Solution [22]

The general solution for a system of $ODEs$ is the solution that contains a number of arbitrary constants equal to the order of the system. It represents all possible solutions. For example, a system of $1^{th} - NLODEs$ contains only one arbitrary constant. But a particular solution for a system of $ODEs$ is the solution obtained from the general solution after substituting specific numerical values for the arbitrary constants (initial conditions or boundary conditions). Sometimes, no solution may be obtained from the general solution, in this case, the solution is called the "singular solution" [10], it is a feature characterizes the system of $NLODEs$. A "singular solution is one that cannot be obtained under any circumstances from the general solution.

## 1.3 The System of $1^{st} - NLODEs$ with Constant Coefficients

### 1.3.1 Definition (The System of $1^{st} - NLODEs$ with Constant coefficients)[25]

The system of $1^{st} - NLODEs$ with Constant coefficients can be written as the following:

$$\frac{dx_i}{dt} = \sum_{i=1}^{n} f_i(x_1, x_2 \dots x_n)\,, i = 1,2,3, \dots n\,, \tag{1.3.1}$$

where $x_i$(t) are dependent variables, $t$ is independent variable and $f_i$ are nonlinear functions (may contains $x_1^3, x_1x_2, sin(x), \dots$). Using the vector form, we can rewrite the system (1.3.1) as the following:

$$\dot{x} = F(x),\ \dot{x}(t) = \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \vdots \\ \dot{x_n}(t) \end{pmatrix}, F(x) = \begin{pmatrix} f_1(x_1, x_2 \dots x_n) \\ f_2(x_1, x_2 \dots x_n) \\ \vdots \\ f_n(x_1, x_2 \dots x_n) \end{pmatrix}. \tag{1.3.2}$$

A system of $1^{st} - NLODEs$ with constant coefficients is a system which consists of a set of $1^{st} - NLODEs$ such that the coefficients are constants do not dependent on the time $t$.

### 1.3.2 The Autonomous and Non-Autonomous System [31]

In an autonomous system there is a set of $1^{st}-ODEs$ in which the independent variable does not appear explicitly, these equations are called autonomous equations, it is describes only the dependent variables. It is on the form (1.3.2), where $t$ is independent variable and $x$ is dependent variable. But a non-autonomous system is a system that every equation in it dependents explicitly on the independent variable, it is on the following form:

$$\dot{x} = f(x, t).$$

For example, the system:

$$\dot{x_1} = -x_2 + x_1^3,$$

$$\dot{x_2} = x_1 + x_2^2,$$

is autonomous nonlinear system. But the system:

$$\dot{x_1} = -x_2 + x_1^3 + sin\ (t),$$

$$\dot{x_2} = x_1 + x_2^2 + e^{-t},$$

is non-autonomous nonlinear system.

The autonomous system widely used in stability, because it allows for the analysis of equilibrium points using Lyapunov functions, which we will discuss in the chapter 3.

### 1.3.3 Solving the Systems of $1^{st}-NLODEs$ with Constant Coefficients[23]

In the following, we give a method to solve the system of $1^{st}-NLODEs$, it is "separate variables". This method is effective for solving systems whose equations can be clearly separated. We will explain this method below.

**Separate Variables Method [12]**

Suppose we have the following system of $1^{st}-NLODEs$:

$$\dot{x_1} = f(x_1), \tag{1.3.3}$$

$$\dot{x_2} = g(x_2). \qquad (1.3.4)$$

To solve this system using separate variables method, we do the following steps:

i. We find the relation between $\dot{x_1}$ and $\dot{x_2}$ by dividing the equations (1.3.3) and (1.3.4), as follows:

$$\frac{\dot{x_2}}{\dot{x_1}} = \frac{g(x_2)}{f(x_1)}.$$

ii. Separate the variables as follows:

$$M(x_2)\dot{x_2} = N(x_1)\dot{x_1}, \qquad (1.3.5)$$

where $M(x_2)$ is a function of the variable $x_2$ only, and $N(x_1)$ is a function of the variable $x_1$ only.

iii. Integrating the equation (1.3.5).
iv. Find relation between $x_1$ and $x_2$ .
v. Find the solutions in terms of $t$ if possible, by referring to the givensystem.

**Example 1.3.3**

We can use the separate variables method to solve the following nonlinear system:

$$\dot{x_1} = x_1x_2, \quad \dot{x_2} = x_2^2,$$

as the following:

*i.* $$\frac{\dot{x_2}}{\dot{x_1}} = \frac{x_2^2}{x_1x_2} = \frac{x_2}{x_1}.$$

*ii.* Separate the variables: $\dot{x_2}x_2 = \dot{x_1}x_1$ .

*iii.* $$\int x_2\dot{x_2} = \int x_1\,\dot{x_1},$$

$$\frac{x_2^2}{2} = \frac{x_1^2}{2} + c,$$

*iv.* The general solution is $x_2^2 - x_1^2 = c$.

**Example 1.3.4**

We can use the separate variables method to sole the following nonlinear system:

$$\dot{x_1} = x_1(1 - x_2^2),$$

$$\dot{x_2} = x_2 x_1^2 \,,$$

as the following:

*i.* $$\frac{\dot{x_2}}{\dot{x_1}} = \frac{x_2 x_1^2}{x_1(1-x_2^2)} = \frac{x_2 x_1}{1-x_2^2} \,.$$

*ii.* Separate the variables

$$x_2 x_1 \dot{x_1} = (1 - x_2^2)\dot{x_2},$$

$$x_1 \dot{x_1} = \frac{1 - x_2^2}{x_2} \dot{x_2},$$

*iii.* $$\int x_1 \dot{x_1} = \int \frac{1-x_2^2}{x_2} \dot{x_2} \,,$$

$$\frac{x_1^2}{2} = ln|x_2| - \frac{x_2^2}{2} + c \,,$$

*iv.* The general solution is:

$$\frac{x_1^2}{2} + \frac{x_2^2}{2} - ln|x_2| = c.$$

**Remark 1.3.1**

One of the most important characteristics of nonlinear systems is the inability to use the superposition principle, that is, if we obtain two or more than two solutions for the nonlinear system, the sum of these solutions is not a solution for this system [19].

# Chapter 2

# Stability of Nonlinear Systems of First Order Ordinary Differential Equations

# Stability of Nonlinear Systems of First Order Ordinary Differential Equations

Stability is of great importance in many fields such as understating the behavior of dynamical systems, ensuring the stability of physical and engineering systems, and mathematical modeling. The importance of stability in $ODE_s$ is complemented by the ability to predict the behavior of the solution, as stability determines whether the solution are close to each other over time or whether they are moving apart with slight changes.

## 2.1 The Equilibrium Point [26]

It is a state of the system space that does not change over time unless an external forces acting on the system are zero. That is, the system stop moving at this point, $\dot{x} = f(x) = 0$. A nonlinear system has more than one equilibrium point. In we have a system $\dot{x} = f(x)$, then the state of the equilibrium point if $x^* = x(0)$, than is $f(x^*) = 0$ for all $t \geq 0$.

We are interested in studing equilibrium points in stability because equilibrium points represent the state f the system, whether it is stable or unstable, and also determine the behavior of the system. Most systems do not remain in a state of motion but tend to seek equilibrium. Equilibrium points contribute to simplifying the analysis of nonlinear systems because the nonlinear system is very complex. By equilibrium, we can generalize aircraft control systems to determine whether the airplane will return to its straight course after small disturbances or lose control and go off course. Equilibrium points are the locations where the system may stabilize [8].

## 2.2 Stability

### 2.2.1 Definition of stability [1]

In mathematics, stability is defined as a state of systems. In other words, it is a mathematics property usually mentioned in connection with the solution of a differential equation, where it is said that the solutions of a differential equation around the equilibrium point have either stable or behavior over time.

### 2.2.2 Types of Stability in Terms of Domain [16]

We can describe the type of stability in terms of domain into two types, they are:

i.Local Stability

When the stability property related to a specific mathematical domain(that is, the solution remains close to the equilibrium point only ) then the beginning is close to the equilibrium point this, the stability is local.

**Example 2.2.1**

If we throw a ball into a small, deep container and move it a little, it will roll to the bottom or stay close to it. This represents local stability.

ii. Global Stability

In this type, the stability property is not related to a specific mathematical domain, which means that the solution tends to words the equilibrium, point no matter how close or far its equilibrium point.

**Example 2.2.2**

If we place a ball at any point on the surface of a large, deep valley or on a mountain that slopes down from all sides, whether this point is far from or close to the point of equilibrium, then this ball will roll towards the bottom, and this represents global stability.

### 2.2.3 Types of Stability in Terms of Temporal behavior [16]

i. **Simple Stability:** It is when the solution remains close to the equilibrium point, but does not necessarily return to it.
ii. **Exponential Stability:** The solution in this type not only remains close, but approaches the equilibrium point at an exponential speed.
iii. **Instability:** In this type, any slight disturbance causes the solution to more away from equilibrium point.
iv. **Asymptotically Stability:** In this type, the solution is close equilibrium point and gradually returns to it over time.

## 2.3 Methods of Finding Equilibrium Point [7]

In a system of $1^{st} - NLODE_s$ , we can find the equilibrium points by means of solutions that make the time rate of change of all the system's variables equal to zero. Since the equilibrium points are the values of *x* that make $\dot{x} = f(x) = 0$, then the only basic and unique method to obtain the equilibrium points is the algebraic method. It is an easy method for simple systems, but it requires complex numerical or analytical methods for more complex systems.

A nonlinear system often has more than one equilibrium point, as illustrated by the following examples.

**Example 2.3.1**

We can find equilibrium points for the following nonlinear system:

$$\dot{x_1} = x_1 - x_1^2 - x_1x_2, \quad (2.3.1)$$

$$\dot{x_2} = 3x_2 - x_1x_2 - 2x_2^2 . \quad (2.3.2)$$

From equation (2.3.1) and (2.3.2) we get:

$$\dot{x_1} = x_1(1 - x_1 - x_2), \quad (2.3.3)$$

$$\dot{x_2} = x_2(3 - x_1 - 2x_2). \quad (2.3.4)$$

The equilibrium point must satisfies that $\dot{x_1} = 0$ and $\dot{x_2} = 0$, so from equations (2.3.3) and (2.3.4) we get:

$$x_1(1 - x_1 - x_2) = 0,$$

$$x_2(3 - x_1 - 2x_2) = 0.$$

From the last two equations:

- If $x_1 = 0$ and $x_2 = 0$, then the equilibrium point $(x_1, x_2) = (0, 0)$.
- If $x_1 = 0$ and $3 - x_1 - 2x_2 = 0$, we get $3 - 0 - 2x_2 = 0$, so $x_2 = 3/2$, then the equilibrium point $(x_1, x_2) = (0, 3/2)$.
- If $1 - x_1 - x_2 = 0$ and $x_2 = 0$, we get $1 - x_1 - 0 = 0$, so $x_1 = 1$, then the equilibrium point $(x_1, x_2) = (1, 0)$.
- If

$$1 - x_1 - x_2 = 0, \tag{2.3.5}$$

$$3 - x_1 - 2x_2 = 0, \tag{2.3.6}$$

from equation (2.3.5), we get, $x_1 = 1 - x_2$. We substitute by $x_1$ into equation (2.3.6) to get $x_2$, as the following:

$$3 - (1 - x_2) - 2x_2 = 0,$$

hence $x_2 = 2$, substitute by $x_2 = 2$ into $x_1 = 1 - x_2$, that is $x_1 = 1 - 2 = -1$, then the equilibrium point is $(x_1, x_2) = (-1, 2)$.

From the above, we get four equilibrium points for the given system, they are: $(0, 0), (0, 3/2), (1, 0), (-1, 2)$.

**Example 2.3.2**

We can find equilibrium points for the following nonlinear system:

$$\dot{x_1} = x_1 + x_2, \tag{2.3.7}$$

$$\dot{x_2} = x_1 x_2 . \tag{2.3.8}$$

The equilibrium point must satisfies that $\dot{x_1} = 0$ and $\dot{x_2} = 0$, so from equations (2.3.7) and (2.3.8) we get:

$$x_1 + x_2 = 0, \tag{2.3.9}$$

$$x_1 x_2 = 0. \tag{2.3.10}$$

By solving the equations (2.3.9) and (2.3.10), we get $x_1 = 0$ and $x_2 = 0$, that is the equilibrium point is $(x_1, x_2) = (0, 0)$.

**2.4 Types of the Equilibrium Point [29]**

i. If the solution are close to the equilibrium point and remain close to it over time, then we say that the equilibrium point is stable.

**Example 2.4.1**

The point to which the ball returns if we move it a little inside a concave container and it returns to the same position, we say that it is a stable point.

ii. If the solutions are close to the equilibrium point and more away from it over time, then we say that the equilibrium point is an unstable.

**Example 2.4.2**

If a ball is at a point at the top of a mountain, and is moved slightly then it will roll away from its position, so the equilibrium point is an unstable point.

iii. If the solution are close to the equilibrium point and do not merely remain close to it over time, but they return to it completely, then we say that the equilibrium point is an asymptotically stable.

**Example 2.4.3**

If a ball moves inside a container and is then slightly displaced inside it, it will move inside the bottom and over time it will stop completely at the bottom, thus are say that the equilibrium point is an asymptotically stable.

iv. If the solution are close to the equilibrium point, such that they are stable on one side, and moving away from the other side over time, then in this case we say that the equilibrium point is a semi stable.

**Example 2.4.4**

If a ball is moved on an inclined surface from one side, then it will roll to words the bottom, and if it is moved from the other side it will move away or falls. In this case we say that the equilibrium point is a semi stable.

The following figure shows types of the equilibrium point:

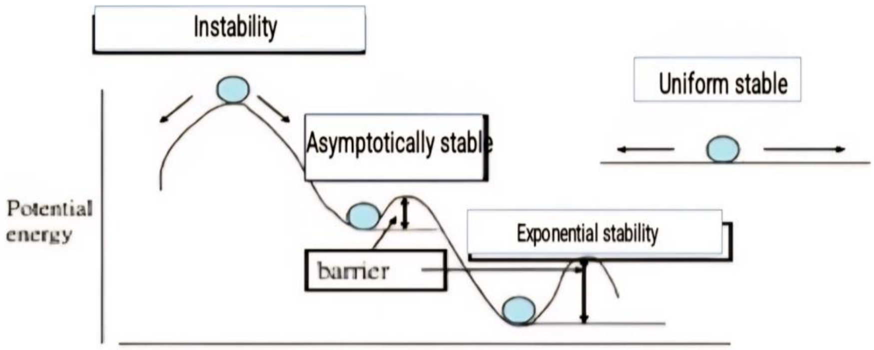


**Figure 2.1 [16] Types of Equilibrium Point**

Equilibrium points are of great importance in stability analysis across many engineering and scientific fields, as they primarily aim to understand the behavior of systems and how changes and disturbances affect them. One of the main reasons we study equilibrium points is to determine whether systems are in a static or stable state and to predict their behavior. In other words, if a system deviates slightly from its equilibrium point after a minor disturbance, will it return to it in a stable state or move further away, becoming unstable? Most stability analyses rely on the system's behavior near equilibrium points, as these points define the zones of stability and instability [2].

## 2.5 Methods for Analyzing the Stability of Equilibrium Points

We can decide whether equilibrium points are stable, unstable, or continentally stable using the following method.

### 2.5.1 Jacobian and the Eigenvalues Method (Linearization) [7]

The Jacobian method is the direct application of indirect Lyapunov method, because it relies on the linearity of the system around the equilibrium point and the judgment of stability, this is done through the eigenvalues of Jacobian matrix. In the following, we explain this method.

Let us consider the following nonlinear two-dimensional system:

$$\left.\begin{aligned} \dot{x_1} &= f_1(x_1, x_2), \\ \dot{x_2} &= f_2(x_1, x_2) \end{aligned}\right\} \qquad (2.5.1)$$

where $x_1, x_2$ are functions of independent variables. Then Jacobian matrix $J(x_1, x_2)$ is defined as follows:

$$J(x_1, x_2) = \begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{pmatrix}. \qquad (2.5.2)$$

We can determine the type of stability of the equilibrium points for the above system using the eigenvalues of Jacobian matrix as follows:

*i.* Find the equilibrium points (see Section (2.3)).
*ii.* Determine the Jacobian matrix $J(x_1, x_2)$.
*iii.* Substitute the equilibrium points in $J(x_1, x_2)$ to obtain a new matrix $J^*$.
*iv.* Find the eigenvalues $\lambda_1$ and $\lambda_2$ of $J^*$ by solving the following characteristic equation:

$$det(J^* - \lambda I) = 0, \qquad (2,5,3)$$

where $det$ denotes to the determinant of the matrix, and $I$ is the identity matrix of size $2 \times 2$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*v.* We can decide if the equilibrium points are stable or unstable or asymptotically stable, through the eigenvalues we obtain, as follows:

1) If $\lambda_1$ and $\lambda_2$are negative real numbers, then the shape of the equilibrium point is a stable node, and the type of stability of the equilibrium point is asymptotically stable, as shown in the following figure:

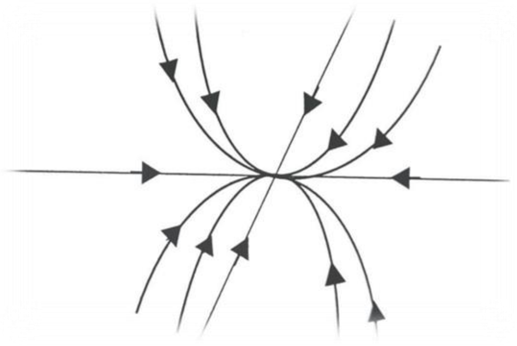

**Figure 2.2 [19] Stable Node**

2) If $\lambda_1$ and $\lambda_2$are two positive real numbers, then the shape of the equilibrium point is an unstable node and the type of stability of the equilibrium point is unstable, as shown in the following figure:

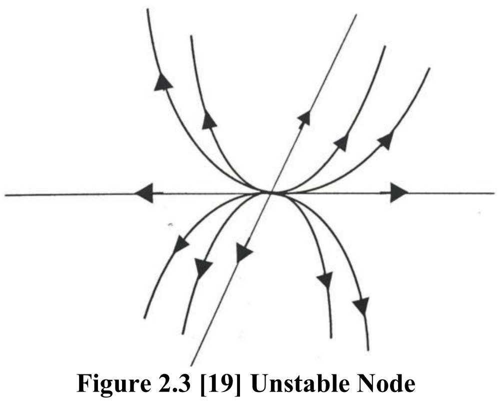

**Figure 2.3 [19] Unstable Node**

3) If $\lambda_1$ and $\lambda_2$are two numbers, one positive and the other negative, then the shape of the equilibrium point is a saddle point and the type of stability of the equilibrium point is unstable.

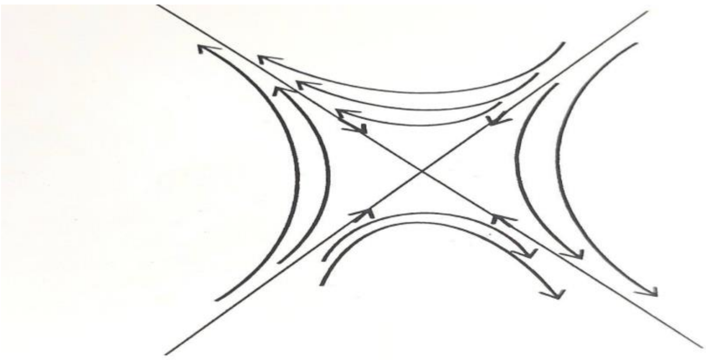

**Figure 2.4 [19] Saddle Point**

4) If $\lambda_1$ and $\lambda_2$are complex numbers, then the shape of the equilibrium point is a stable focus, and the type of stability of the equilibrium point is asymptotic stability.

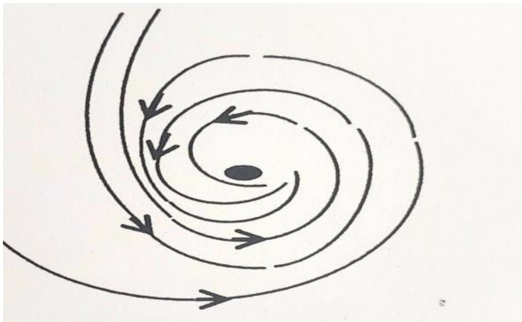

**Figure 2.5 [19] Stable Focus**

5) If $\lambda_1$ and $\lambda_2$are complex numbers $(a \pm ib, a > 0)$, then the shape of the equilibrium point is unstable focus, and the type of stability of the equilibrium point is unstable.

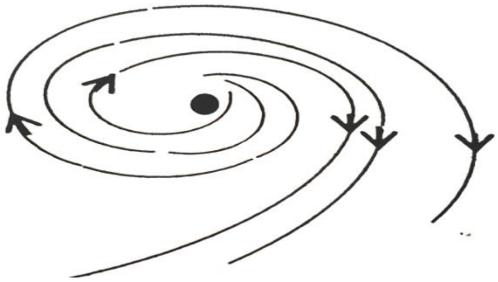

**Figure 2.6 [19] Unstable Focus**

6) If $\lambda_1$ and $\lambda_2$are complex numbers, then the shape of the equilibrium point is centre point, and the type of stability of the equilibrium point is stable.

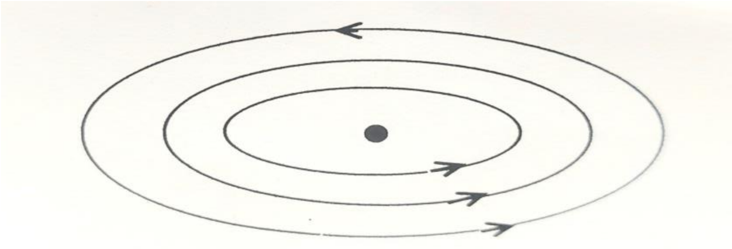

**Figure 2.7 [19] Centre Point**

7) If $\lambda_1$ and $\lambda_2$ are two positive real numbers such that $\lambda_1 = \lambda_2 > 0$, then the shape of the equilibrium point is an unstable node and the type of stability of the equilibrium point is unstable.

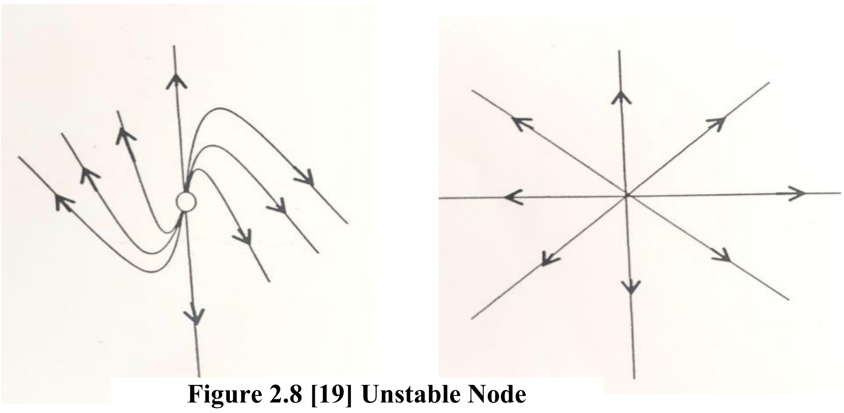

**Figure 2.8 [19] Unstable Node**

8) If $\lambda_1$ and $\lambda_2$are two positive real numbers such that $\lambda_1 = \lambda_2 > 0$ then the shape of the equilibrium point is a stable node and the type of stability of the equilibrium point is stable.

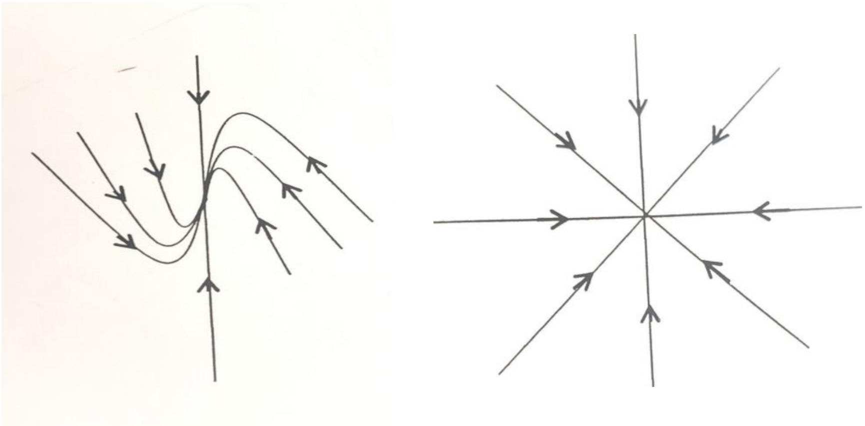

**Figure 2.9 [19] Stable Node**

### Remark 2.5.1

In the case of $\lambda_1 = 0$ or $\lambda_2 = 0$, we cannot determine the type of stability of the equilibrium point, so the Jacobi method in this case is not sufficient to determine the type of stability of the equilibrium point. Therefore, we need the Lyapunov method to determine the type of stability, and this is what we will talk about in detail in Chapter 3.

### Example 2.5.1

We can study the type of stability of the equilibrium points for the following nonlinear system:

$$\dot{x_1} = x_1 - x_1^2 - x_1 x_2, \tag{2.5.4}$$

$$\dot{x_2} = -x_2 + x_1 x_2 \,, \tag{2.5.5}$$

using the eigenvalues of Jacobian matrix as follows:

1) We find the equilibrium points. To do that we put $\dot{x_1} = 0$ and $\dot{x_2} = 0$ in the equations (2.5.4) and (2.5.5), we get:

$$x_1 - x_1^2 - x_1 x_2 = 0,$$

$$-x_2 + x_1 x_2 = 0,$$

we can rewrite the last equations as follows:

$$x_1(1 - x_1 - x_2) = 0, \tag{2.5.6}$$

$$x_2(-1 + x_1) = 0, \tag{2.5.7}$$

From the equations (2.5.6) and (2.5.7), we have:

- If $x_1 = 0$ and $x_2 = 0$, then the equilibrium point $(x_1, x_2) = (0, 0)$.
- If $x_1 = 0$ and $-1 + x_1 = 0$, we get $x_1 = 1$, this is impossible.
- If $1 - x_1 - x_2 = 0$ and $x_2 = 0$, we gett $1 - x_1 - 0 = 0$, so $x_1 = 1$, so the equilibrium point $(x_1, x_2) = (1, 0)$.

From the above, the equilibrium points of the system are $(0, 0), (1, 0)$, both appearing repeatedly in the solutions.

2) Determine the Jacobian matrix using the form (2.5.2), where $(x_1, x_2) = (0, 0)$, we get:

$$J(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 - x_2 & x_1 \\ x_2 & -1 + x_1 \end{pmatrix},$$

3) $$J^* = J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J^{**} = J(1,0) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

4) Find the eigenvalues of $J^*$ by solving the following characteristic equation:

$$det(J^* - \lambda I) = det\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = det\begin{pmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{pmatrix} = 0,$$

$$(1 - \lambda)(-1 - \lambda) = 0,$$

$$1 - \lambda = 0, \ \lambda = 1,$$

$$-1 - \lambda = 0, \lambda = -1.$$

Now, we find the eigenvalues of $J^{**}$ by solving the following characteristic equation:

$$det(J^{**} - \mu I) = det\left(\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\right) = det\begin{pmatrix} -1 - \mu & 1 \\ 0 & -\mu \end{pmatrix} = 0,$$

$$-\mu(-1 - \mu) = 0,$$

$$\mu = 0, \ \mu = -1,$$

5) Note that $\lambda = 1$ and $\lambda = -1$, that is, one positive and the other negative, then the shape of the equilibrium point is a saddle point and the type of stability of the equilibrium point (0, 0) is unstable. Also, note that $\mu = 0$ and $\mu = -1$, in this case we cannot determine the type of stability of the equilibrium point.

### 2.5.2 Phase Plane Method [27]

In systems of $1^{st} - NLODEs$, we can effectively use the phase plane to study the type of stability graphically rather than finding explicit analytical solutions to these 1zsystems, as finding such analytical solutions is difficult or impossible in these systems. The phase plane is a graphical tool for understanding the overall behavior of systems more descriptive than demonstrative. In other words, it is very important in qualitative analysis for understanding the behavior of solutions, meaning it determines the nature of the point and its path behavior as a geometric representation of the solutions; it is a visual geometric behavior. This method is explained in the following steps.

Let us consider the nonlinear system on the form (2.5.1).

1) Find the equilibrium points for the system (2.5.1).
2) Draw the vectors $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$.
3) The shape of the paths around the equilibrium point illustrates the type of stability of the equilibrium point, that is:

- Paths enter to the point from all directions, in this case the type of stability of equilibrium point is stable.
- Paths exit from the equilibrium point or move away from it, in this case the type of stability of equilibrium point is unstable.
- Paths enter from one direction and exit from another, in this case the type of stability of equilibrium point is asymptotically stable.

The following figure illustrate that.

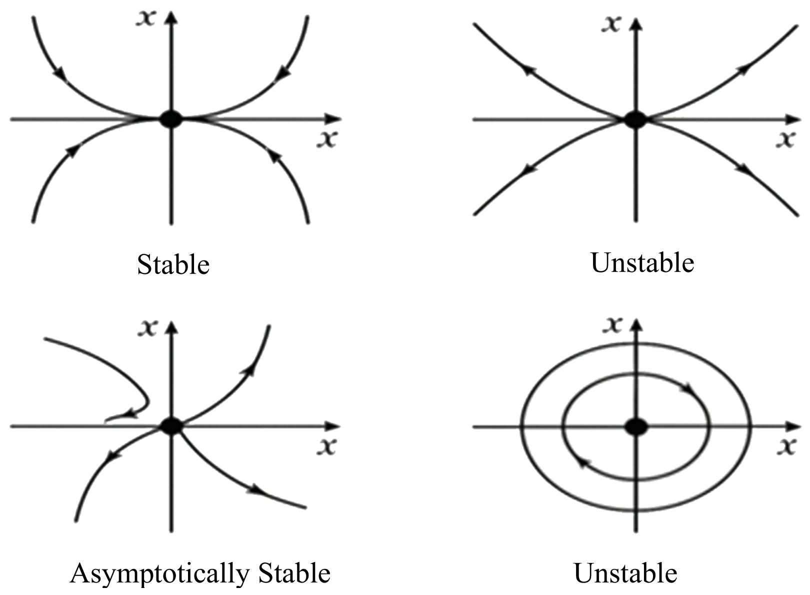


**Figure 2.10 [27] Classification of Equilibrium Points Using Phase Plane**

**Example 2.5.2**

We can study the type of stability of the equilibrium points for the following nonlinear system:

$$\dot{x_1} = -x_1 - x_2^2,$$

$$\dot{x_2} = -x_2\ ,$$

using the phase plane method, as follows:

1) We put $\dot{x_1} = 0$ and $\dot{x_2} = 0$ in the given system to find the equilibrium points, as follows:

$$-x_1 - x_2^2 = 0, \qquad (2.5.10)$$

$$-x_2 = 0. \qquad (2.5.11)$$

From (2.5.11) we get $x_2 = 0$, substituting in equation (2.5.10), we get $x_1 = 0$, so the equilibrium point is (0, 0).

2) Draw the first vector $-x_1 - x_2^2$ by finding points as follows:

Choose arbitrary value for $x_1$, such that it is greater than zero, where $x_2 = 0$. For example $x_1$ =0.1, so $-x_1 - x_2^2 = -(0.1) - 0 = -0.1$. hence the new point is $(-0.1, 0)$. Choose another arbitrary value for $x_1$, such that it is less than zero, where $x_2 = 0$. For example $x_1 = -0.1$, so $-x_1 - x_2^2 = -(-0.1) - 0 = 0.1$, So the new point is $(0.1, 0)$.

Now, we draw the second vector $-x_2$ by the same away. Choose arbitrary value for $x_2$, such that it is greater than zero, where $x_1 = 0$. For example $x_2$ =0.1, so the new point is $(0, 0.1)$. Choose another arbitrary value for $x_2$, such that it is less than zero, where $x_1 = 0$. For example $x_2 = -0.1$, so the new point is $(0, -0.1)$.

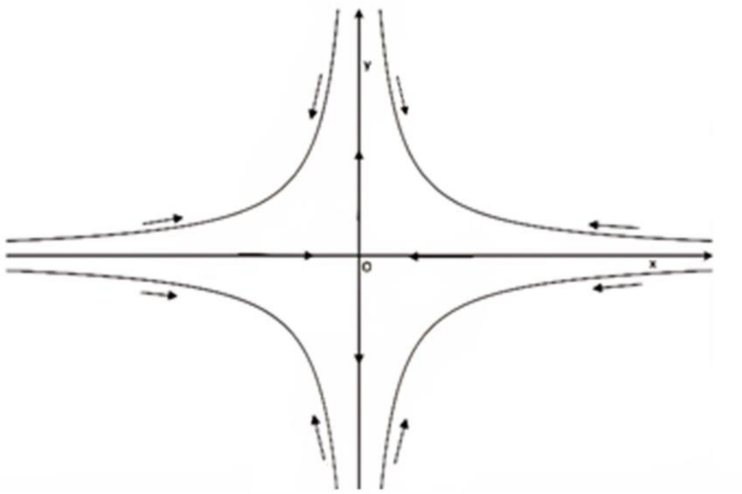


**Figure 2.11 [27] Illustration of Example (2.5.2)**

From the above figure, we note the equilibrium point is (0, 0) is stable because all paths go to this point.

### 2.5.3 Lyapunov Method [15]

This method considers from the important methods to study types of stability. It depends on building a suitable function called Lyapunov function, which will be discussed in detail in Chapter 3.

# Chapter 3

# Using Lyapunov Functions to Study the Stability of Nonlinear Systems of First Order Ordinary Differential Equations

# Using Lyapunov Functions to Study the Stability of Nonlinear Systems of First Order Ordinary Differential Equations with Constant Coefficients

In 1892, the Russian scientist Alexander Lyapunov published his doctoral dissertation, "The General Problem of Stability of Motion", which contained many fruitful ideas and important results. These results made it possible to divide the study of stability into two periods: the pre- Lyapunov period and the post-Lyapunov period. Lyapunov provided a precise definition of the stability of motion, in addition to presenting two fundamental methods for analyzing stability problems. He studied the concepts of stability by finding solutions that were applied concisely and led to significant results.

Lyapunov worked on deriving the stability properties of the equilibrium of a system described by a nonlinear equation from the stability properties of its linearity (its transformation into a linear equation). This method is called Lyapunov indirect method or Lyapunov first method. However, he developed a more efficient method, Lyapunov direct method, which does not rely on prior knowledge of the solutions but deals directly with differential systems using special auxiliary functions called Lyapunov functions, which will be the subject of this chapter. Lyapunov worked on this method for over 100 years, and it became the primary tool for dealing with stability problems in various types of equations. It is also known for its efficiency and simplicity.

## 3.1 Basic Concepts and Definitions

Let us consider the nonlinear system $\dot{x} = f(x)$, for study the stability of this system, we give the following definition.

**Definition 3.1.1 [2]**

The equilibrium point According to Lyapunov is the point at which the system comes to a complete stop. In other words, it is the point where all derivatives of the system are equal to zero, such that if the system starts from this point, it will always remain there.

To find the equilibrium point, we put all derivatives $\dot{x}$ equal to zero, then solve the equations of the system. The point where $f(x^*) = 0$ is the equilibrium point. Here, we chose the equilibrium point at (0,0), because it simplifies the analysis, doesn't affect the result, and preserves generality.

Lyapunov did not invent a new point of equilibrium, but rather he invented the language and mathematical tools necessary to understand the behavior of the system near this point and hoe its stability is secured.

**Definition 3.1.2 [21]**

The equilibrium point $x^*$ is said to be stable according to Lyapunov if the following condition is satisfied:

$\forall \varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq 0$. (3.1.1)

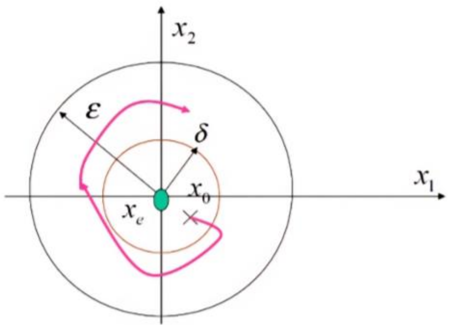


**Figure 3.1 [16] Stability in Lyapunov Concept**

**Definition 3.1.3 [21]**

Instability according to Lyapunov is defined as the negation of the stability condition, i.e., if the condition (3.1.1) does not satisfy, then the equilibrium point is unstable.

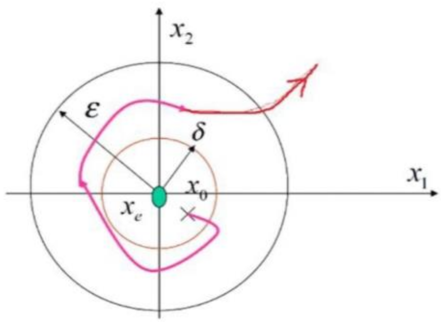


**Figure 3. 2 [16] Instability in Lyapunov Concept**

**Definition 3.1.4 [21]**

The equilibrium point $x^*$ is asymptotically stable if it is stable according to Lyapunov, in addition to the presence of an attractive region around it, such that if the movement starts within this region, it will converge towards equilibrium, i.e.,

$$\lim_{t\to\infty} \|x(t) - x^*\| = 0.$$

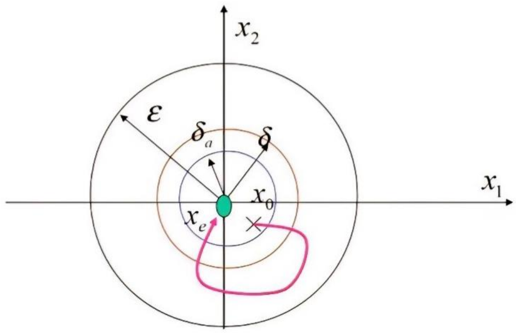


**Figure 3.3 [16] Asymptotically Stable in Lyapunov Concept**

## 3.2 Types of Stability According to Lyapunov [11]

*i.* Uniform stability: It is a type of stability where $\delta$ and $\varepsilon$ do not depend on the initial time $t$, meaning that this type of stability is not affected by changes in the initial time, it is uniform over time.

*ii.* Exponential stability: This is a type of stability where the solution not only stays close to the equilibrium point, but approaches it at an exponential speed over time. That is, as time increases, the distance from the equilibrium point decreases at an exponential speed, i.e., at a large and regular speed.

*iii.* Instability: It is the opposite of stability, meaning that if the course of the situation or solution starts very close to the point of equilibrium, the solution moves away from that point over time.

## 3.3 The Conditions of Lyapunov Function [28]

Lyapunov's theorem is fundamental to analyzing the behavior of dynamical systems, especially nonlinear systems. It relies on selecting a function that resembles the energy of the system. This function must be positive everywhere except at the equilibrium point. The Lyapunov function method is used to directly investigate the stability of the equilibrium position of a system $\dot{x} = f(x)$ with help of suitably chosen function $V(x)$ (the Lyapunov function). This is done without finding the solutions for the systems. In the following we give conditions of Lyapunov function $V(x)$.

1) $V(x)$ must be specific definitely positive: meaning, $V(x) > 0$, for all $x^* = x, V(x^*) = 0$. That is, the energy is always positive and becomes zero only at the equilibrium point [13].

2) $V(x)$ must be continuous and its partial derivatives must also be continuous.

**3.4 The Lyapunov Function and its The Importance [10]**

The scientist Lyapunov demonstrated that certain functions could be used for stability analysis instead of analysis. These functions are called Lyapunov functions; they are natural functions and energy functions, and they are the tools for applying Lyapunov theory to a specific system. Lyapunov functions are standard functions, denoted by the symbol v, which we choose to analyze the stability of a system. They enable us to understand the system's behavior and avoid the difficulty of solving the problem.

Choosing a Lyapunov function is a fundamental and influential step in analyzing the stability of systems, as there is no fixed rule that predetermines the form of the function. Rather, the choice depends on the nature of the system under study.

i. If the system is physical, we choose an energy function.

ii. If the system is algebraic, we use a general quadratic function and adjust the coefficients.

iii. If there is no clear method, we use linearization or trial and error. Since choosing the appropriate Lyapunov function for any system, especially in a nonlinear system, is the most difficult and the most important step in stability analysis, the Lyapunov function chosen in this research was chosen to be in the following form:

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2). \quad (3.4.1)$$

This function is the most used function in stability analysis, because it is always strictly positive (determined) for every $(x_1, x_2) \neq (0,0)$, and it is zero only at the equilibrium point, its derivatives with respect to time are easy to calculate, and it helps determine the type of stability using Lyapunov conditions without needing to solve the system completely. Also, the equilibrium point is the origin because most dynamical systems can be transformed so that the equilibrium point is the origin without losing generality. The importance of Lyapunov functions lies in their being a powerful tool for understanding the behavior of systems without needing to solve differential equations directly. They can be used in all linear and nonlinear systems and do not depend on linear properties or eigenvalues. From the sign of the derivative of the function $V(x_1, x_2)$, we can decide the type of stability [3].

The derivative of the function $V(x_1, x_2)$ is $\dot{V}(x_1, x_2)$, it takes the following form:

$$\dot{V}(x_1,x_2) = \dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x)$$

$$= \begin{pmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \frac{\partial V}{\partial x} f(x). \tag{3.4.2}$$

In general if $V(x)$ is Lyapunov function in the variables $x_1, x_2, \dots, x_n$, then:

$$\dot{V}(x_1, x_2, \dots, x_n) = \dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x_\iota} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

$$= \begin{pmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} = \frac{\partial V}{\partial x} f(x).$$

**3.5 Basic Theorems**

In the following, we give important theorems which help us to study the stability of nonlinear systems of $1^{st} - NLODEs$.

**Theorem 3.5.1 [7]**

The function $V(x_1, x_2) = x_1^2 + ax_1x_2 + bx_2^2$ is:

*i.* positive definitely iff $4b^2 - a^2 > 0$.

*ii.* Semi-positive iff $4b^2 - a^2 \geq 0$.

**Theorem 3.5.2 [7]**

Suppose that $V(x_1, x_2)$ is Lyapunov function for the following system:

$$\dot{x_1} = F(x_1, x_2),$$
$$\dot{x_2} = G(x_1, x_2).$$

*i.* If $\dot{V}(x_1, x_2)$ negative-semi definitely, then the origin is stable.

*ii.* If $\dot{V}(x_1, x_2)$ definitely negative, then the origin is asymptotically stable.

*iii.* If $\dot{V}(x_1, x_2)$ definitely positive, then the origin is unstable.

**Remarks 3.5.1**

i. $V(x)$ is definitely negative, meaning $V(x) < 0$ for all $x \neq 0$, i.e., the function is less than zero at all points except the origin, $V(0) = 0$.

ii. $V(x)$ is semi-negative, meaning $V(x) \leq 0$ for all $x$, $V(0) = 0$.

**iii.** $V(x)$ is semi-positive, meaning $V(x) \geq 0$ for all $x$ in the vicinity, $V(0) = 0$ at the equilibrium point.

**Theorem 3.5.3 [11] (Lyapunov Theorem for Asymptotic Stability)**

If we have a system of differential equations $\dot{x} = f(x)$, and a constant-signed function $V(x)$ such that its total derivative with respect to time is also a constant-signed function but with the opposite sign to $V(x)$, then the equilibrium point $x^* = 0$ is asymptotically stable.

### 3.6 Lyapunov Method for Studing the Stability

We explain Lyapunov method in the following steps:

1) Write the nonlinear system $\dot{x} = f(x)$ about the equilibrium point.
2) Choose a Lyapunov function $V(x)$, such that always be positive, i.e., $V(x) > 0$, for all $x \neq 0$. We choose this function such that it satisfies the condition $V(0) = 0$.
3) Calculate the Lyapunov derivative (3.4.2) along the paths of the system.
4) We analyze the sign of the derivative:
   - *i.* If $\dot{V}(x) < 0$, the system is asymptotically stable.
   - *ii.* If $\dot{V}(x) \leq 0$, the system is stable according to Lyapunov.
   - *iii.* If $\dot{V}(x) > 0$, the system is unstable.
5) Interpret the result: We deduce the type of stability based on the previous conditions without needing to solve the system.

### 3.7 Illustrative Examples

**Example 3.7.1**

Investigate the stability of the equilibrium point of the following system

$$\dot{x_1} = (-x_1)^3 + x_1 {x_2}^2,$$

$$\dot{x_2} = -2{x_1}^2 x_2 - {x_2}^3.$$

The equilibrium point of the given system is the origin (0,0). We suggest Lyapunov function (3.4.1), it is:

$$V(x_1, x_2\,) = \frac{1}{2}(x_1^2 + x_2^2).$$

Note that: $V(0, 0\,) = 0$. We use the equation (3.4.2), as follows:

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2}$$

$$= x_1(-{x_1}^3 + x_1{x_2}^2) + x_2(-2{x_1}^2 x_2 - {x_2}^3)$$

$$= -{x_1}^4 + {x_1}^2{x_2}^2 - 2{x_1}^2{x_2}^2 - {x_2}^4$$

$$\dot{V}(x_1,x_2) = -{x_1}^4 - {x_1}^2{x_2}^2 - {x_2}^4 < 0,$$

Since $V(x_1,x_2)$ is definitely positive (using theorem (3.5.1)), and $\dot{V}(x_1,x_2) < 0$, then the equilibrium point is asymptotically stable.

**Example 3.7.2**

Investigate the stability of the equilibrium point of the following system

$$\dot{x_1} = {x_1}^3,$$

$$\dot{x_2} = 2{x_1}^2x_2 + 4{x_1}^2x_2 + 2x_2.$$

The equilibrium point of the given system is the origin (0,0). We suggest Lyapunov function (3.4.1), it is:

$$V(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

$$V(0,0) > 0.$$

We use the equation (3.4.2), as follows:

$$\dot{V}(x_1,x_2) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2}$$

$$= x_1({x_1}^3) + x_2(2{x_1}^2x_2 + 4{x_1}^2x_2 + 2x_2)$$

$$= {x_1}^4 + 2{x_1}^2{x_2}^2 + 4{x_1}^2{x_2}^2 + 2{x_2}^2$$

$$= {x_1}^4 + 6{x_1}^2{x_2}^2 + 2{x_2}^2$$

$$\dot{V}(x_1,x_2) = {x_1}^4 + 6{x_1}^2{x_2}^2 + 2{x_2}^2 > 0,$$

Since $V(x_1,x_2)$ is definitely positive (see theorem (3.5.1)), and $\dot{V}(x_1,x_2) > 0$, then the equilibrium point is unstable.

**Example 3.7.3**

Investigate the stability of the equilibrium point of the following system

$$\dot{x_1} = -x_1x_2,$$

$$\dot{x_2} = {x_1}^2 - {x_2}^3.$$

The equilibrium point of the given system is the origin (0,0). We suggest Lyapunov function (3.4.1), it is:

$$V(x_1, x_2\ ) = \frac{1}{2}(x_1^2 + x_2^2).$$

We test the equilibrium point: $V(0, 0\ ) = 0$. We use the equation (3.4.2), as follows:

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2}$$

$$\dot{V}(x_1, x_2) = x_1(-x_1 x_2) + x_2({x_1}^2 - {x_2}^3)$$

$$= -{x_1}^2 x_2 + {x_1}^2 x_2 - {x_2}^4$$

$$= -{x_2}^4$$

$$\dot{V}(x_1, x_2) = -{x_2}^4 \leq 0,$$

Since $V(x_1, x_2\ )$ is definitely positive (see theorem (3.5.1)), and $\dot{V}(x_1, x_2) \leq 0$, then the equilibrium point is stable.

# Chapter 4

# Application of Stability Analysis Using Lyapunov Functions on a 3D-Drone

# Application of Stability Analysis on a 3D-Drone

In this chapter, we apply our study in the third chapter to analysis the stability of a drone in three-dimensional space (3D-drone) using a nonlinear mathematical model. This model incorporates translational motion along the three axes, as well as the rotational angles around each axis.

We begin by constructing a comprehensive physical model that accounts for the forces generated by the four propellers, gravity, and moments (torques). The model assumes that all system parameters such as mass and moment of inertia—remain constant. Subsequently, this model is converted into a system of $1^{st} - NLODEs$. Following this, we determine the principal equilibrium point, defined by zero velocities, zero angles, and balanced thrust force.

Accordingly, Lyapunov stability theory is applied to analyze the system's stability. A quadratic Lyapunov function, based on all system states, is tested to verify stability [4-6, 16, 17, 26, 30].

**Figure 4.1 3D-Drone**

## 4.1 Basic Concepts

In this section, we give important concepts needed to understand our work in this chapter.

**Definition 4.1.1 (Moment)**

The Moment is the effect of rotation caused by a force acting on an object around a specific point or axis.

**Definition 4.1.2 (Position Vector)**

A position vector is a vector that describes the position of a point relative to a fixed reference point (often the origin), it starts at the reference point and ends at the point whose position is to be described, specifying its direction and distance from the reference point. It is a fundamental concept in physics and engineering for accurately describing motion and spatial relationships. It is denoted by the symbol $p$.

**Definition 4.1.3 (Acceleration of Gravity)**

The acceleration of gravity is the acceleration that freely falling objects gain near the Earth's surface and it is approximately $9.8\ meters/second^2 (9.8\ m/s^2)$. It is denoted by the symbol $g$, that is $g = 9.8\ m/s^2$ .

**Definition 4.1.4 (Angular Velocity)**

The angular velocity is a measure of how fast an object rotates about an axis, or the amount of angular distance the object travels per unit time, measured in Radians per second $(rad/s)$.

**Definition 4.1.5 (Rotation Matrix)**

A rotation matrix is a square orthogonal matrix used in linear algebra to transform vector or coordinate systems in Euclidean space without changing the length of the vector or the size of the shape while keeping the axes constant. It is used to represent direction in some fields, such as robotics, image equations, and computers: It denoted by $R$.

**Definition 4.1.6 (Euler's constant)**

Euler's constant is a fundamental mathematical constant, approximately equal to 2.71828, and is used in analysis, differential, and integration as a basis for exponential functions.

**Definition 4.1.7 (Euler's angles)**

Euler's angles are a set of three consecutive intrinsic rotations that describe the orientation of a three- dimensional object. They are widely used in physics, engineering and mechanics to describe the rotation motion of a three- dimensional object, such as the rotation of an airplane or a spacecraft.

**Definition 4.1.8 (Resistance )**

Resistance is a physical property of metallic conductors in electrical circuits. It is defined as the ability of materials to resist electric current. It is denoted by $F_d$.

**Definition 4.1.9 (Linear Velocity)**

Linear velocity is the rate of change of an object's position with respect to time during its movement in a straight line, it is measured in meter/second. In other words, the linear velocity is a vector quantity that expresses both the speed and direction of an object's motion at a given instant, it is denoted by $v$.

**Definition 4.1.10 (The Perturbed Moment)**

The perturbed moment is the effect of disturbance on physical quantities, such as dipole moment in batteries and electricity, or torque in mechanics due to the presence of an external field or force, which causes a change in the behavior of the basic system, it is denoted by $\tau_{dist}$.

### 4.1.11 Control Moment

Control moment is a rotational force used to guide and stabilize mechanical systems such as satellites and robots, it is implemented via a Gyroscope mechanisms. It is denoted by $\tau$.

### 4.1.12 Moment Matrix

A moment matrix is a matrix that describes how mass is distributed in a rigid body with respect to a given axis of rotation. It defines the relationship between angular velocity, and between moment and angular acceleration in three dimensions. It is denoted by $I$ .

### 4.1.13 Positive Matrix

The positive matrix is a comprehensive square matrix that makes the value of the square shape always positive for any non-zero vector. It is denoted by $k$.

### 4.1.14 Mass

The mass is a measure of the amount of matter in an object, it is a constant property that does not change with location or gravity. It is also known as a measure of inertia (an object's resistance to change in its state of motion), it is denoted by $m$ and measured in gram or kilogram

## 4.2 Model Equations for a 3D-Drone

$$m\ddot{p} = mge_{\varepsilon} + R(\eta)\begin{pmatrix} 0 \\ 0 \\ \tau \end{pmatrix} + F_d(v), \tag{4.2.1}$$

where

$$e_{\varepsilon} = (0 \quad 0 \quad -1)^t = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

$p$ is a position vector, its components are called position components. $\dot{p}$ is a linear velocity, $\eta$ is a vector of Euler's angles, $\omega$ is a vector of angular velocities:

$$p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \dot{p} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix},\ \eta = \begin{pmatrix} \varphi \\ \theta \\ \psi \end{pmatrix}, \omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.$$

Now, if we consider the direction of gravity to be downwards and $F_d(v)$ represents the effect of air resistance, then we write:

$$\dot{p} = v,$$

$$\dot{v} = \frac{1}{m} R(\eta) e_\varepsilon T + g e_\varepsilon + \frac{1}{m} F_d(v). \qquad (4.2.2)$$

$$g e_\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix},$$

is chosen according to the direction.

Now, the rotational equations:

$$I\dot{\omega} + \omega(I\omega) = \tau + \tau_{dist}\ , \qquad (4.2.3)$$

where $\tau_{dist}$ represents the disturbance moments that can be neglected in fundamental analysis, also

$$\dot{\eta} = J(\eta)\omega, \qquad (4.2.4)$$

so, we define the following vector:

$$x = \begin{pmatrix} p \\ v \\ \eta \\ \omega \end{pmatrix},$$

to get the following standard formula:

$$\dot{x} = f(x) + G(x)U,$$

where $f(x)$ sums the coefficients of gravity and internal nonlinearity, $G(x)$ represents the distribution of the input over the equations.

Now, we choose the equilibrium point:

$$p_{eq} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},$$

where $x_0 = y_0 = z_0$ , so:

$$v_{eq} = 0, \eta_{eq} = \begin{pmatrix} 0 \\ 0 \\ \psi_0 \end{pmatrix}, \psi_0 = 0, \omega_{eq} = 0, T_{eq} = mg, \tau_{eq} = 0.$$

Now, we define:

$$\delta x = x - x_{eq} ,$$

$$\delta U = U - U_{eq} ,$$

so,

$$\delta \dot{x} = A \cdot \delta x + B\delta U,$$

where

$$A = \frac{\partial f}{\partial x}, B = \frac{\partial f + GU}{\partial U} = G(x),$$

the matrices $A$ and $B$ are constant at the equilibrium point, they use to design a linear controller and prove stability using Lyapunov function.

Now, we choose two rings:

i. The outer ring:

We position error locator:

$$e_p = p - p_{ref} \ ,$$

and error velocity:

$$e_v = v - v_{ref} \ ,$$

then, we choose:

$$ad_{e_\varepsilon} = -k_p e_p - k_v e_p + a_{ref} \ ,$$

where $k_p$ and $k_v$ are positive matrices. Now, we want to verify the following equation:

$$\frac{1}{m} R(\eta) e_\varepsilon T + g e_\varepsilon = a_{ref} \ ,$$

therefore, we need to pay:

$$T_{cmd} = m\|a_{des} - g e_\varepsilon\|.$$

ii. The inner ring

We derive the error direction as the following:

$$e_R = \frac{1}{2}(R_{des}^t R - R^t R_{des}),$$

$$\tau = -k_R e_R - k_\omega(\omega - \omega_{des}) + \omega(I\omega),$$

where $k_R$ and $k_\omega$ are positive matrices.

### 4.3 Choosing Lyapunov Function

We choose the following Lyapunov function:

$$V(x) = \frac{1}{2} m \left\|v - v_{ref}\right\|^2 + \frac{1}{2}\left(p - p_{ref}\right)^t k_p \left(p - p_{ref}\right) + \frac{1}{2}\omega^t I \omega + \psi(R, R_{des}). \qquad (4.2.5)$$

Lyapunov function was chosen for an energy similar to the sum of the kinetic energy and the squares of positional errors., such that $\psi(R, R_{des})$ is a directional error measurement function.

**Properties of Lyapunov selected function**

i. $V(x)$ is a positive function.
ii. $V(x) = 0$ at $\omega = 0$.
iii. $R = R_{des}$ , $v = v_{ref}$ , $p = p_{ref}$ .
iv. $\dot{V}$ depends on $v, a_{des}$ and $\tau$. $T$ $and$ $\tau$ can be selected such that $\dot{V}$ is negative.

Now, we calculate $\dot{V}$ as the following:

Firstly, we calculate the derivative of $V(x)$ with the equation (4.2.5) along the path, part by part. We derive the first two terms of the equation (4.2.5), as the following:

$$\frac{d}{dt}\left(\frac{1}{2}m\|v - v_{ref}\|^2\right) = m\left(v - v_{ref}\right)^t \dot{v} \,,$$

$$\frac{d}{dt}\left(\frac{1}{2}\left(p - p_{ref}\right)^t k_p\left(p - p_{ref}\right)\right) = \left(p - p_{ref}\right)^t k_p\left(v - v_{ref}\right).$$

Secondly, we derive the parts of the rotation as the following:

$$\frac{d}{dt}\left(\frac{1}{2}\omega^t I\omega\right) = \omega^t(I\dot{\omega}),$$

$$\frac{d}{dt}\left(\psi(R, R_{des})\right) = (e_R)k_R(\omega - \omega_{des}).$$

Now, we collect all parts and substitute $\dot{v}$ with its equivalent in equation (4.2.2), and $\dot{\omega}$ with its equivalent in equation (4.2.3), we get:

$$\dot{V}(x) = (p - p_{ref})^t k_p(v - v_{ref}) + m(v - v_{ref})^t \left(\frac{1}{m}R(\eta)e_\varepsilon T + ge_\varepsilon + \frac{1}{m}F_d(v)\right) + \omega^t(\tau - \omega(I\omega)) + (e_R)k_R(\omega - \omega_{des}).$$

Finally, we organize the functions to get:

$$\dot{V} \leq -c_1 \left\|v - v_{ref}\right\|^2 - c_2 \left\|e_R\right\|^2 - c_3 \left\|\omega\right\|^2,$$

where $c_i > 0, i = 1, 2, 3$. Since $V(x)$ is positive, and the time limit $\dot{V} \leq 0$, then the path remains confined within the surface area of the plane $V(x)$.

The fundamental result is that the stability of the 3D-drone can be comprehensively guaranteed in three-dimensional space.

### 4.4 Numerical Simulation

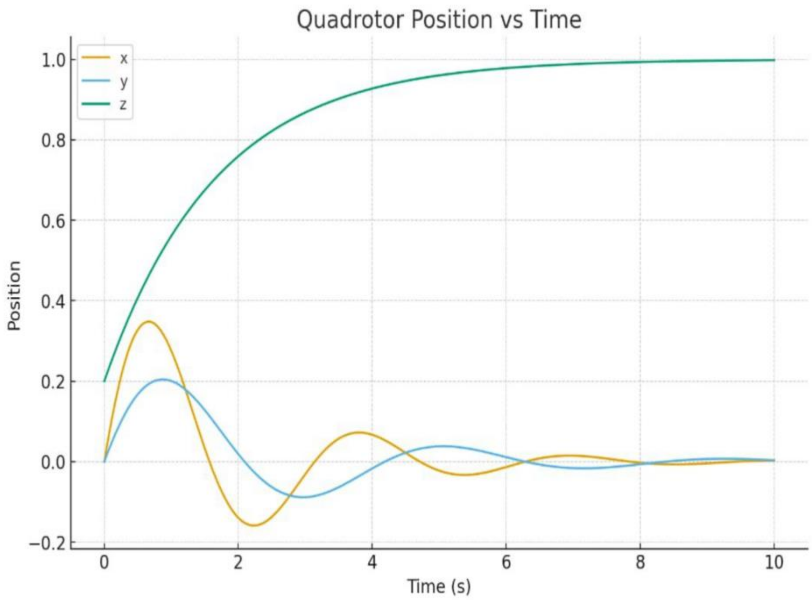


**Figure 4.2 Quadrotor Position**

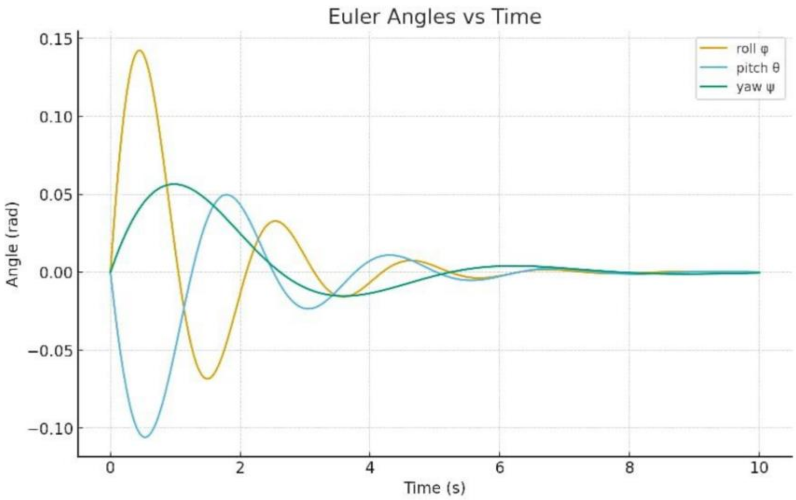


**Figure 4.3 Euler Angles**

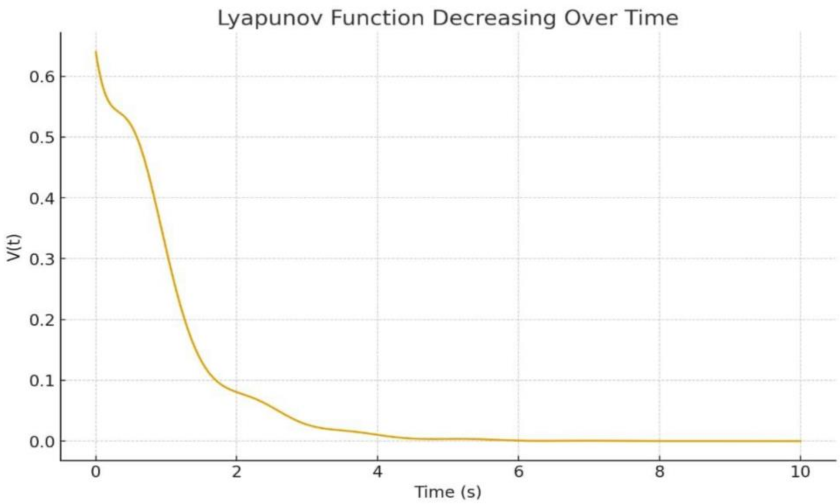


**Figure 4.4 Lyapunov Function**

# Conclusion

Based on the stability analysis conducted in our study, which included a review of systems of $1^{st} - NLODEs$, the concept of equilibrium points, and some methods of stability analysis "Jacobian and the eigenvalue, path plane", culminating in the direct Lyapunov method, in addition to an applied model "3D drone motion modeling", we get the following results:

i. The effectiveness and limitations of traditional methods in determining the type of stability around the equilibrium points of simple systems, and that Jacobian method is an actual application of the indirect Lyapunov method.

ii. Classical methods, such as the phase plane approach and the Jacobian method, mainly rely on geometric representations or on linearizing the system around an equilibrium point. Consequently, their results are often limited in scope and do not accurately capture the full nonlinear behavior of the system. In contrast, Lyapunov's direct method is a rigorous analytical tool that is based on the original nonlinear model without requiring explicit solutions of the differential equations. This approach provides a clear mathematical framework for proving stability, making it particularly well suited for the analysis of complex nonlinear systems.

iii. Lyapunov function (3.4.1) was chosen because it is simple form greatly facilitated the differential process, allowing for a focus on the behavior of the system rather than the mathematical complexity of the complex Lyapunov function.

iv. Lyapunov functions are the ideal choice to ensure the stability of complex dynamic systems such as aircraft, provided that the obstacle of choosing the appropriate function is overcome.

v. The practical application to 3D-drone led to the success of the modeling, meaning that it was determined that the movement of the 3D-drone could be accurately formulated as a nonlinear dynamic system.

# Recommendations for Future Research

We recommend for future studying, studing and analysis the stability of nonlinear systems of fractional order ordinary differential equations with variable coefficients using Lyapunov methods, also analysis the stability of nonlinear systems with time-delays and using the Mathematica (or Matlab) program to construct Lyapunov functions symbolically and numerically.

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# Appendix 1

```
%simulate_quad3D.m
محاكاة %Quadrotor 3D مع تحكم position->attitude و حساب V و Vdot
Clear; clc; close all;

---------------- %PARAMETERS----------------
M = 1.2;                  % mass (kg)
I = diag([0.014, 0.014, 0.028]); % Inertia matrix (kg*m^2)
G = 9.81;

%aerodynamic damping (simple linear drag)
D = diag([0.1, 0.1, 0.2]);

%controller gains (tune these)
Kp_pos = diag([6,6,10]);     % position P
Kv_pos = diag([4.5,4.5,6]);  % velocity D

Kp_att = diag([80,80,60]);   % attitude P (for phi,theta,psi)
Kw_att = diag([4,4,1.5]);    % angular rate D

%reference (hover at origin, z_ref = 1.0 m)
P_ref = [0;0;1.0];
V_ref = [0;0;0];
Psi_ref = 0; % desired yaw
```

```
%initial condition: [p; v; eta (phi theta psi); omega]
X0 = [  0.5; -0.4; 0.8;    % position x,y,z
%            ;0 ;0 ;0         linear velocities
%     ;0.0 ;0.15 ;0.1         phi, theta, psi (rad)
%    ;[ 0.0 ;0.0 ;0.0         angular rates p,q,r

Tspan = [0 10];

%ODE solve
Opts = odeset('RelTol',1e-6,'AbsTol',1e-8);
[t, X] = ode45(@(t,X) quad3D_dynamics(t, X, m, I, g, D, Kp_pos, Kv_pos, Kp_att, Kw_att,
p_ref, v_ref, psi_ref), tspan, X0, opts);

%compute control Inputs, V and Vdot along trajectory
N = length(t);
T_traj = zeros(N,1);
Tau_traj = zeros(N,3);
V_traj = zeros(N,1);
Vdot_traj = zeros(N,1);
For i=1:N
    Xi = X(I, :)';
    [Tcmd, tau, phi_des, theta_des, psi_des, a_des] = controller_quad3D(Xi, m, g, Kp_pos,
Kv_pos, Kp_att, Kw_att, p_ref, v_ref, psi_ref);
    T_traj(i) = Tcmd;
    Tau_traj(I,:) = tau';
    [V_traj(i), Vdot_traj(I)] = computeLyap_quad3D(Xi, m, I, Kp_pos, Kp_att, p_ref, v_ref,
psi_ref, Tcmd, tau);
```

```
End

 ---------------- %PLOTS----------------

3 %D trajectory

Figure;

Plot3(X(:,1), X(:,2), X(:,3), 'LineWidth', 1.8); hold on;

Plot3(p_ref(1), p_ref(2), p_ref(3), 'r*','MarkerSize',10);

Xlabel('x (m)'); ylabel('y (m)'); zlabel('z (m)');

Grid on; title('3D Trajectory'); view(45,30);

 %positions

Figure;

Subplot(3,1,1); plot(t, X(:,1)); ylabel('x (m)'); grid on;

Subplot(3,1,2); plot(t, X(:,2)); ylabel('y (m)'); grid on;

Subplot(3,1,3); plot(t, X(:,3)); ylabel('z (m)'); xlabel('t (s)'); grid on;

Sgtitle('Positions');

 %Euler angles

Figure;

Subplot(3,1,1); plot(t, X(:,7)); ylabel('\phi (rad)'); grid on;

Subplot(3,1,2); plot(t, X(:,8)); ylabel('\theta (rad)'); grid on;

Subplot(3,1,3); plot(t, X(:,9)); ylabel('\psi (rad)'); xlabel('t (s)'); grid on;

Sgtitle('Euler angles');

 %linear and angular velocities

Figure;
```

```
Subplot(2,1,1);

Plot(t, X(:,4), t, X(:,5), t, X(:,6)); legend('v_x','v_y','v_z'); title('Linear velocities'); grid on;

Subplot(2,1,2);

Plot(t, X(:,10), t, X(:,11), t, X(:,12)); legend('p','q','r'); title('Angular rates'); grid on;


 %Lyapunov V and Vdot

Figure;

Subplot(2,1,1); plot(t, V_traj, 'LineWidth',1.5); title('V(t)'); grid on;

Subplot(2,1,2); plot(t, Vdot_traj, 'LineWidth',1.5); title('Vdot(t)'); grid on;

Xlabel('t (s)');


 %control inputs

Figure;

Subplot(2,1,1); plot(t, T_traj, 'LineWidth',1.2); title('Total Thrust T'); grid on;

Subplot(2,1,2);  plot(t,  tau_traj);  legend('\tau_x','\tau_y','\tau_z');  title('Torques  \tau');
grid on;


 %Print final state & V

Fprintf('Final position: [%.4f, %.4f, %.4f]\n', X(end,1), X(end,2), X(end,3));

Fprintf('Final Euler (phi,theta,psi): [%.4f, %.4f, %.4f]\n', X(end,7), X(end,8), X(end,9));

Fprintf('Final V = %.6f, Vdot = %.6f\n', V_traj(end), Vdot_traj(end));
```

# Appendix 2

## Construction of Equations

```
Function dX = quad3D_dynamics(~, X, m, I, g, D, Kp_pos, Kv_pos, Kp_att, Kw_att, p_ref,
v_ref, psi_ref)

 %X = [p(3); v(3); eta(3); omega(3)]


 %unpack

P = X(1:3);

V = X(4:6);

Phi = X(7); theta = X(8); psi = X(9);

Omega = X(10:12);


 %controller -> compute T and tau

[Tcmd, tau, phi_des, theta_des, psi_des, a_des] = controller_quad3D(X, m, g, Kp_pos,
Kv_pos, Kp_att, Kw_att, p_ref, v_ref, psi_ref);


 %rotation matrix R (body to inertial) from ZYX Euler: R = Rz(psi)*Ry(theta)*Rx(phi)

R = rotationMatrixFromEuler(phi, theta, psi);


 %translational dynamics

 %gravity vector (in inertial): [0;0;-g]

Grav = [0;0;-g];


 %drag

Fd = - D * v;
```

```
V_dot = (1/m) * (R * [0;0;Tcmd]) + grav + (1/m)*Fd;

 %rotational dynamics
Omega_dot = I \ (tau – cross(omega, I*omega));

 %Euler rates mapping from body rates omega to euler_dot
Teta = eulerRatesMatrix(phi, theta);
Eta_dot = Teta * omega;

 %pack derivative
dX = zeros(12,1);
dX(1:3) = v;
dX(4:6) = v_dot;
dX(7:9) = eta_dot;
dX(10:12) = omega_dot;
end

 ---- %helper functions----
Function R = rotationMatrixFromEuler(phi, theta, psi)
    Rz = [ cos(psi) -sin(psi) 0; sin(psi) cos(psi) 0; 0 0 1];
    Ry = [ cos(theta) 0 sin(theta); 0 1 0; -sin(theta) 0 cos(theta)];
    Rx = [1 0 0; 0 cos(phi) -sin(phi); 0 sin(phi) cos(phi)];
    R = Rz * Ry * Rx;
End

Function T = eulerRatesMatrix(phi, theta)
```

```
%    maps body rates [p;q;r] to Euler angle rates [phi_dot; theta_dot; psi_dot]
%    [phi_dot; theta_dot; psi_dot] = T * [p; q; r]
   T = [ 1, sin(phi)*tan(theta), cos(phi)*tan(theta);
,0          cos(phi),          -sin(phi);
,0          sin(phi)/cos(theta), cos(phi)/cos(theta);[
End
```

# Appendix 3

# Calculation Lyapunov Function

```
Function [V, Vdot] = computeLyap_quad3D(X, m, I, Kp_pos, Kp_att, p_ref, v_ref, psi_ref,
Tcmd, tau)

 %X: current state

P = X(1:3); v = X(4:6);

Phi = X(7); theta = X(8); psi = X(9);

Omega = X(10:12);


 %errors

E_p = p – p_ref;

E_v = v – v_ref;

E_att = [phi; theta; psi] - [0;0;psi_ref];


 %Lyapunov candidate (energy-like)

V = 0.5*m*(e_v'*e_v) + 0.5*e_p' * Kp_pos * e_p + 0.5 * omega' * I * omega + 0.5 *
e_att' * Kp_att * e_att;


 %compute Vdot numerically using gradient * f (approx)

 %Build grad V components

dV_dp = Kp_pos * e_p;

dV_dv = m * e_v;

dV_datt = Kp_att * e_att;

dV_domega = I * omega;
```

```
%compute state derivative f (same as in dynamics but without needing full function call)

%We need p_dot = v

P_dot = v;


%get R and gravity vector (use rotation as In dynamics)

R = rotationMatrixFromEuler(phi, theta, psi);

Grav = [0;0;-9.81];


%approximate drag ignored here (since we don't have D here) - acceptable for Vdot sign check

V_dot = (1/m) * (R * [0;0;Tcmd]) + grav;


%omega_dot approx from tau and I (neglecting cross term for simplicity here)

Omega_dot = I \ (tau – cross(omega, I*omega));


%e_att_dot approximate via Euler rates matrix

Teta = eulerRatesMatrix(phi, theta);

Att_dot = Teta * omega;


%form f vector

F = [p_dot; v_dot; att_dot; omega_dot];


%gradient vector In state ordering [p; v; att; omega]

gradV = [dV_dp; dV_dv; dV_datt; dV_domega];


Vdot = gradV' * f; % scalar
```

```
End

 %helper functions (repeat)

Function R = rotationMatrixFromEuler(phi, theta, psi)

    Rz = [ cos(psi) -sin(psi) 0; sin(psi) cos(psi) 0; 0 0 1];

    Ry = [ cos(theta) 0 sin(theta); 0 1 0; -sin(theta) 0 cos(theta)];

    Rx = [1 0 0; 0 cos(phi) -sin(phi); 0 sin(phi) cos(phi)];

    R = Rz * Ry * Rx;

End

Function T = eulerRatesMatrix(phi, theta)

    T = [ 1, sin(phi)*tan(theta), cos(phi)*tan(theta);

 ,0          cos(phi),          -sin(phi);

 ,0          sin(phi)/cos(theta), cos(phi)/cos(theta);[

End
```

# الملخص

في هذا العمل ندرس استقرار الأنظمة غير الخطية من المعادلات التفاضلية العادية من الرتبة الأولى بمعاملات ثابتة بإستخدام دوال ليبانوف. لذلك ندرس استقرار هذه الأنظمة بشكل عام، ثم نقدم بعض المفاهيم الأساسية بخصوص نقطة التوازن وأنواع الاستقرار بمعنى ليبانوف، بالإضافة إلى بعض نظريات ليبانوف التي تساعدنا في تحليل الاستقرار بإستخدام دوال ليبانوف. باستخدام هذه المنهجية تمكننا من دراسة سلوك النظام بدون إيجاد حلول صريحة للنظام. لقد طبقنا هذه المنهجية لتحليل استقرار طائرة بدون طيار في الفضاء ثلاثي الأبعاد (3D-drone) باستخدام نموذج غير خطي يتضمن الحركة الانتقالية على طول المحاور الثلاثة بالإضافة إلى زوايا الدوران حول كل محور. وضحنا كل ذلك بأمثلة.

مدرسة العلوم الأساسية

قسم الرياضيات

# دراسة استقرار أنظمة المعادلات التفاضلية العادية غير الخطية باستخدام دوال ليبانوف

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